The complete form of moment equations of stellar dynamics

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Abstract

The exact mathematical expression for an arbitrary n^{th}-order stellar hydrodynamic equation is explicitly obtained depending on the central moments of the velocity distribution. In such a form the equations are physically meaningful, since they can be compared with the ordinary hydrodynamic equations of compressible, viscous fluids. The equations are deduced without any particular assumptions about symmetries, steadiness, or particular kinematic behaviours, so that they can be used in their complete form, and for any order, in future works with improved observational data. Also, in order to work with a finite number of equations and unknowns, which would provide a dynamic model for the stellar system, the n^{th}-order equation is needed to investigate in a more general way the closure conditions, which may be expressed in terms of velocity distribution statistics, as it is shown in a case example.

KEY WORDS: hydrodynamics – methods: analytical – stars: kinematics – galaxies: kinematics and dynamics – galaxies: statistics.

1991 MATHEMATICS SUBJECT CLASSIFICATION: 60, 62, 85.

1 Introduction

The stellar hydrodynamic equations have been used in a number of works on galactic dynamics to study the stellar mass and velocity distributions, either from an analytical viewpoint (e.g. Vandervoort 1975, Hunter 1979, Evans & Lynden-Bell 1989, Evans et al. 2000, van de Ven et al. 2003) or as a model for numerical simulations to investigate the shape of the velocity distribution, or to reproduce the spiral structure of galactic disks as an alternative way to the N-body approach (e.g. Korchagin et al. 2000, Orlova et al. 2002, Vorobyov & Theis 2006). However, only equations of mass, momentum and, in few cases, energy transfer are generally handled, and, in most cases, axial symmetry, steady-state stellar system, and other hypotheses are assumed. There are few works that, in a mathematical aspect, have gone beyond such a basic assumptions. Sala et al. (1985) proposed a general expression for the n^{th}-order equation, without steadiness and axisymmetry, although it was written depending on the absolute, non-comoving moments of the stellar velocity distribution, where, by substitution of the moments as a series of the pressures, they obtained a general but non-explicit expression of the equations. The explicit equations were, in the end, specifically written for orders $n = 0, 1, 2, 3, 4$. However, it is well known that stellar hydrodynamic equations are physically meaningful when they can be compared with the ordinary hydrodynamic equations of a compressible, viscous fluid, and this is only possible when they are written in terms of the tensors of comoving moments or pressures, in the reference frame associated with the local centroid. Often, these expansions or computational procedures are provided instead of their explicit expression, and they are later used to simplify and to close the system of equations, for example to study a cool, pure rotating disc (Aoki 1985, Amendt & Cuddeford 1991). On the other hand, the work by Cuddeford & Amendt (1991) had also a general and more interesting mathematical scope, although it was restricted to steady-state systems, amid other hypotheses. They studied higher-order stellar hydrodynamic equations, by using central velocity moments up to eighth-order, and they investigated some quite general conditions over the velocity distribution in order to close the infinite hierarchy of the moment equations.

However, actual kinematic data, e.g. from *HIPPARCOS* catalogue (ESA 1997), do not support any more the hypotheses of axisymmetry, steadiness, or pure galactic rotation (e.g. Cubarsi & Alcobé 2006). In addition, the future *GAIA* mission (Katz et al. 2004, Wilkinson et al. 2005) will represent a major improvement with respect to *HIPPARCOS*, since the three velocity components will be available for the largest number of stars ever collected, where an unbiased radial velocity component will provide essential information to kinematic and dynamic studies of the Galaxy.

The aim of the present work is to provide the exact and full mathematical expression for an arbitrary n^{th}-order equation depending on the pressures, or alternatively on the comoving moments, without any additional hypotheses. Such a general expression should be taken as a starting point in forthcoming works, either to use improved observational data, or to carry out more exhaustive numerical simulations. In addition, the exact n^{th}-order equation is also essential to study more general closure conditions, or under less restrictive assumptions, for building up more accurate dynamical models from a finite number of equations and variables.

Let us introduce the notation by reviewing some basic concepts. From a macroscopic approach, a stellar system is described by giving their distribution in the phase space, which consists in couples of three-dimensional vectors $\mathbf{r}$ and $\mathbf{V}$ representing star position and velocity, measured from an inertial reference system. The stellar distribution is then given through the phase space density function $f(t, \mathbf{r}, \mathbf{V})$, which is assumed as continuous and differentiable in nearly every point, providing, at time t, the number of stars with position within the range $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$, and velocity between $\mathbf{V}$ and $\mathbf{V} + d\mathbf{V}$.

It is generally assumed that the Galaxy is at present in a state in which each star can be idealised as a conservative dynamical system to a very high degree of accuracy. Thus, the forces acting in the system can be associated with a gravitational potential function per unit mass $\mathcal{U}(t, \mathbf{r})$, so that the motion of a star is described in a Cartesian coordinates system by the equations

$$\dot{\mathbf{r}} = \mathbf{V}, \quad \dot{\mathbf{V}} = -\mathcal{U}(t, \mathbf{r}) \tag{1}$$

This is a Hamiltonian system, where Liouville theorem is satisfied. That is, the density of any element of phase space remains constant during its motion. Therefore f satisfies, by using the Stokes operator $\frac{D}{Dt}$, the following equation,

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \mathbf{V} \cdot \nabla_{\mathbf{r}} f + \dot{\mathbf{V}} \cdot \nabla_{\mathbf{V}} f = 0 \tag{2}$$

The foregoing relationship is usually referred as collisionless Boltzmann equation (Hénon 1982), or fundamental equation of stellar dynamics. It is well known that the right hand side of Eq. 2 is in general non-null, but it contains a term giving account of the collisions, $\left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}}$, which would provide the complete form of Liouville equation. However, the collisional relaxation time is long in large stellar systems. The time of relaxation for stellar encounters in the solar neighbourhood is greater than 10^{13} years (Binney & Tremaine 1987), while the galactic rotation period is about 10^8 years. Hence, the encounters are entirely unimportant. The collisions can not be neglected in a globular cluster which contain 10^5 stars, but for a galaxy of 10^{11} stars, the relaxation time turns out to be much larger than the age of the universe, and the encounters can be neglected.
The collisionless Boltzmann equation may be regarded from two different viewpoints. By substitution of $\dot{\mathbf{V}}$ from Eq. 1 into Eq. 2, the equation stands for

$$\frac{\partial f}{\partial t} + \mathbf{V} \cdot \nabla_{\mathbf{r}} f - \nabla_{\mathbf{r}} \mathcal{U} \cdot \nabla_{\mathbf{V}} f = 0 \tag{3}$$

so that it is either a linear and homogeneous partial differential equation for f, for a given potential $\mathcal{U}$, or a linear non-homogeneous partial differential equation for $\mathcal{U}$, where the density function f is already known. Both approaches have been largely studied since Eddington (1921) and Oort (1928), and among other works, those of Vandervoort (1979), de Zeeuw & Lynden-Bell (1985), Bienaymé (1999) and Famaey et al. (2002) may be pointed out.
Obviously, neither the phase space density function nor the potential are observable quantities, while we do have enough large data sets of the full space motions in the solar neighbourhood for different types of stars, like those derived from the *HIPPARCOS* catalogue, or those hopefully forthcoming from the *GAIA* mission, in order to compute the kinematic statistics of the distribution. Then, in order to isolate information about the spatial properties of the stellar system, the collisionless Boltzmann equation can be integrated over the velocity space, or in a more general way, it may be multiplied through by any powers of the velocities before integrating, and each choice of powers leads to a different equation which involves the kinematic statistics describing the stellar system for fixed time and position, which are the mean velocity and the moments of the velocity distribution. Such a strategy, which is usually referred as moment or fluid approach, provides us with an infinite hierarchy of stellar hydrodynamic equations, which could be used as a dynamical model to study the stellar system, on condition that some closure relationships were available in order to work with a finite number of equations and unknowns.
Thus, and since as far as I know the exact and complete n^{th}-order equation, explicitly depending on the comoving moments, has not been yet published, it will be obtained according to the following steps. After reviewing in the next section the statistics describing the stellar system, in §3 the

collisionless Boltzmann equation will be integrated in the space of peculiar velocities. The resulting equation will be obtained depending on some auxiliary tensors $\mathbf{Q}_n$, which give account of the fraction of the transferred pressures $\mathbf{P}_n$ as a linear function of the velocity gradient and of the force due to relative stress variations. In §4, the relationship between $\mathbf{Q}_n$ and $\mathbf{P}_n$ will be established, so that the stellar hydrodynamic equations can be expressed in terms of the pressures. In §5, the n^{th}-order stellar hydrodynamic equation will be explicitly written depending on the comoving moments of the velocity distribution. In §6 a simple example will show how the obtained general expression can be used to study the closure conditions from which the hydrodynamic equations and the collisionless Boltzmann equation are equivalent. Finally, in §7, some concluding remarks will be presented.

2 Velocity distribution

For fixed values of time t and position $\mathbf{r}$ the macroscopic properties of a stellar system can be described from the following statistics, which provide indirect information of the phase space density function. The stellar density is given by

$$N(t,\mathbf{r}) = \int_{\mathbf{V}} f(t,\mathbf{r},\mathbf{V})\, d\mathbf{V} \tag{4}$$

and the stellar mean velocity, or velocity of the centroid, is

$$\mathbf{v}(t,\mathbf{r}) = \frac{1}{N(t,\mathbf{r})} \int_{\mathbf{V}} \mathbf{V}\, f(t,\mathbf{r},\mathbf{V})\, d\mathbf{V} \tag{5}$$

If the peculiar velocity of a star is denoted by

$$\mathbf{u} = \mathbf{V} - \mathbf{v}(t,\mathbf{r}) \tag{6}$$

then the symmetric tensor of the n^{th}-order central moments is obtained from the following expected value

$$\mathbf{M}_n(t,\mathbf{r}) = E[(\mathbf{u})^n] = \frac{1}{N(t,\mathbf{r})} \int_{\mathbf{V}} (\mathbf{V} - \mathbf{v}(t,\mathbf{r}))^n\, f(t,\mathbf{r},\mathbf{V})\, d\mathbf{V}, \quad n \geq 0 \tag{7}$$

where $(\cdot)^n$ stands for the n^{th}-tensor power. The tensor $\mathbf{M}_n$ has $\binom{n+2}{2}$ different elements according to the expression

$$\mu_{\alpha_1\alpha_2\ldots\alpha_n} = E[u_{\alpha_1} u_{\alpha_2} \ldots u_{\alpha_n}] \tag{8}$$

so that the indices belong to the set $\{1,2,3\}$, depending on the velocity component. Obviously, $\mathbf{M}_0 = 1$, $\mathbf{M}_1 = \mathbf{0}$.

The tensor of the central moments is equivalent to the tensor of temperatures from the kinetic theory of gases, while the tensor of pressures is given by

$$\mathbf{P}_n = N\, \mathbf{M}_n \tag{9}$$

Thus, in order to introduce the kinematic statistics into the collisionless Boltzmann equation, Eq. 2 is multiplied by the n^{th}-tensor power of the star velocity and then integrated over the whole velocity space,

$$\int_{\mathbf{V}} (\mathbf{V})^n \, \frac{Df}{Dt} \, d\mathbf{V} = (\mathbf{0})^n, \quad n \geq 0 \tag{10}$$

where, in the integration process, since there are not stars with infinite velocities, the following boundary conditions are assumed,

$$\lim_{|\mathbf{V}| \to \infty} (\mathbf{V})^n f(t, \mathbf{r}, \mathbf{V}) = (\mathbf{0})^n, \; n \geq 0 \tag{11}$$

For each value of n, the tensor equation Eq. 10 leads to the n^{th}-order stellar hydrodynamic equation, which provides us with a conservation law *along the centroid trajectory*. The most basic cases are the continuity equation, for $n = 0$, which stands for mass conservation, and the momentum conservation equation, for $n = 1$.

However, the methodology of most books on galactic dynamics which devote any chapter to obtain or discuss the stellar hydrodynamic equations (e.g. Chandrasekhar 1942, Kurth 1957, Ogorodnikov 1965, Mihalas 1968, Binney & Tremaine 1987), is to integrate Eq. 10 –for $n = 0$ and $n = 1$– over the absolute, non-peculiar velocities, leading to equations involving the absolute moments of the velocity distribution, and afterwards, to give a physical interpretation of each equation, the total moments are explicitly written in function of the central moments. Such a procedure is appropriate for the lowest order equations, but it is not adequate for an arbitrary n^{th}-order equation. The consequence is that, to my knowledge, the general expression for such an arbitrary order equation is nowhere published.

3 Hydrodynamic equations

Let us write the collisionless Boltzmann equation, Eq. 2, in terms of the stellar mean velocity, Eq. 5, and of the peculiar velocities, Eq. 6, by expressing the phase space density function f in the form

$$\phi(t, \mathbf{r}, \mathbf{u}) = f(t, \mathbf{r}, \mathbf{u} + \mathbf{v}(t, \mathbf{r})) \tag{12}$$

where t, $\mathbf{r}$ and $\mathbf{u}$ are independent variables. Hence, the derivatives with respect to these variables will be

$$\begin{gathered}
\frac{\partial f}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{\partial \mathbf{u}}{\partial t} \cdot \nabla_{\mathbf{u}} \phi = \frac{\partial \phi}{\partial t} - \frac{\partial \mathbf{v}}{\partial t} \cdot \nabla_{\mathbf{u}} \phi \\
\nabla_{\mathbf{r}} f = \nabla_{\mathbf{r}} \phi + \nabla_{\mathbf{r}} \mathbf{u} \cdot \nabla_{\mathbf{u}} \phi = \nabla_{\mathbf{r}} \phi - \nabla_{\mathbf{r}} \mathbf{v} \cdot \nabla_{\mathbf{u}} \phi \\
\nabla_{\mathbf{V}} f = \nabla_{\mathbf{V}} \mathbf{u} \cdot \nabla_{\mathbf{u}} \phi = \nabla_{\mathbf{u}} \phi
\end{gathered} \tag{13}$$

Then Eq. 2 becomes

$$\frac{\partial \phi}{\partial t} - \frac{\partial \mathbf{v}}{\partial t} \cdot \nabla_{\mathbf{u}} \phi + (\mathbf{u} + \mathbf{v}) \cdot (\nabla_{\mathbf{r}} \phi - \nabla_{\mathbf{r}} \mathbf{v} \cdot \nabla_{\mathbf{u}} \phi) - \nabla_{\mathbf{r}} \mathcal{U} \cdot \nabla_{\mathbf{u}} \phi = 0 \tag{14}$$

so that, by reorganising terms, it yields

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \phi - \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{v} \right) \cdot \nabla_{\mathbf{u}} \phi + \mathbf{u} \cdot (\nabla_{\mathbf{r}} \phi - \nabla_{\mathbf{r}} \mathbf{v} \cdot \nabla_{\mathbf{u}} \phi) - \nabla_{\mathbf{r}} \mathcal{U} \cdot \nabla_{\mathbf{u}} \phi = 0 \tag{15}$$

Now, to simplify the notation, we shall use the material derivative associated with the centroid motion,

$$\frac{d}{dt}(\cdot) = \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\right)(\cdot) \tag{16}$$

Since $\mathbf{r}$ and $\mathbf{u}$ are independent variables, we shall use the identity

$$\mathbf{u}\cdot\nabla_{\mathbf{r}}\phi = \nabla_{\mathbf{r}}\cdot(\mathbf{u}\,\phi) \tag{17}$$

and we shall also consider the following equality[1]

$$\mathbf{u}\cdot\nabla_{\mathbf{r}}\mathbf{v}\cdot\nabla_{\mathbf{u}}\phi = \nabla_{\mathbf{r}}\mathbf{v} : (\mathbf{u}\otimes\nabla_{\mathbf{u}}\phi) \tag{18}$$

where each dot represents an inner product, and $\otimes$ a tensor product[2]. Notice that the colon stands for the dot products $\nabla_{\mathbf{r}}$ with $\mathbf{u}$, and $\mathbf{v}$ with $\nabla_{\mathbf{u}}$, respectively.
Hence, Eq. 15 may be written as follows

$$\frac{d\phi}{dt} - \left(\frac{d\mathbf{v}}{dt} + \nabla_{\mathbf{r}}\mathcal{U}\right)\cdot\nabla_{\mathbf{u}}\phi + \nabla_{\mathbf{r}}\cdot(\mathbf{u}\phi) - \nabla_{\mathbf{r}}\mathbf{v} : (\mathbf{u}\otimes\nabla_{\mathbf{u}}\phi) = 0 \tag{19}$$

We take now the tensor product of the foregoing equation with the n^{th}-tensor power of the peculiar velocity $(\mathbf{u})^n$,

$$(\mathbf{u})^n\frac{d\phi}{dt} - (\mathbf{u})^n\otimes\left[\left(\frac{d\mathbf{v}}{dt} + \nabla_{\mathbf{r}}\mathcal{U}\right)\cdot\nabla_{\mathbf{u}}\phi\right] + \nabla_{\mathbf{r}}\cdot\left[(\mathbf{u})^{n+1}\phi\right] - \nabla_{\mathbf{r}}\mathbf{v} : \left[(\mathbf{u})^{n+1}\otimes\nabla_{\mathbf{u}}\phi\right] = (\mathbf{0})^n \tag{20}$$

and the resulting equation is then integrated over the peculiar velocities, where the factors depending only on $\mathbf{r}$ and t are left out of the integrals. Thus, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbf{u}}(\mathbf{u})^n\,\phi\,d\mathbf{u} - \left(\frac{d\mathbf{v}}{dt} + \nabla_{\mathbf{r}}\mathcal{U}\right)\cdot\int_{\mathbf{u}}(\mathbf{u})^n\otimes\nabla_{\mathbf{u}}\phi\,d\mathbf{u} + \\ &+\nabla_{\mathbf{r}}\cdot\int_{\mathbf{u}}(\mathbf{u})^{n+1}\,\phi\,d\mathbf{u} - \nabla_{\mathbf{r}}\mathbf{v} : \int_{\mathbf{u}}(\mathbf{u})^{n+1}\otimes\nabla_{\mathbf{u}}\phi\,d\mathbf{u} = (\mathbf{0})^n \end{aligned} \tag{21}$$

The first and third terms of the above relationship are directly expressed in function of the pressures, according to Eq. 7 and Eq. 9. Instead, for the other terms an auxiliary tensor may be defined as follows

$$\mathbf{Q}_{n+1} = -\int_{\mathbf{u}}(\mathbf{u})^n\otimes\nabla_{\mathbf{u}}\phi\,d\mathbf{u},\ n\geq 0 \tag{22}$$

so that Eq. 21 may be re-written in a more compact notation,

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[1] In component notation the equality can be written as $u_i\dfrac{\partial v_j}{\partial r_i}\dfrac{\partial\phi}{\partial u_j} = \dfrac{\partial v_j}{\partial r_i}\left(u_i\dfrac{\partial\phi}{\partial u_j}\right)$, where Einstein's summation criterion for repeated indices is applied.

[2] The notation used for nabla operators is the usual one. If $\mathbf{x}$ is a vector variable and $\mathbf{F}_n(\mathbf{x})$ a n-rank symmetric tensor field, then for $n\geq 1$ the divergence $\nabla_{\mathbf{x}}\cdot\mathbf{F}_n$ is, in components, $\frac{\partial}{\partial x_{i_1}}F_{i_1\ldots i_n}$, while for $n\geq 0$, $\nabla_{\mathbf{x}}\mathbf{F}_n$ is used instead of $\nabla_{\mathbf{x}}\otimes\mathbf{F}_n$ to represent the gradient $\frac{\partial}{\partial x_{i_1}}F_{i_2\ldots i_{n+1}}$.

$$\frac{d\mathbf{P}_n}{dt} + \left(\frac{d\mathbf{v}}{dt} + \nabla_{\mathbf{r}}\mathcal{U}\right) \cdot \mathbf{Q}_{n+1} + \nabla_{\mathbf{r}} \cdot \mathbf{P}_{n+1} + \nabla_{\mathbf{r}}\mathbf{v} : \mathbf{Q}_{n+2} = (\mathbf{0})^n \tag{23}$$

However, the tensors $\mathbf{Q}_n$ are not directly computable in their current form.

4 Conservation of pressures

The next step is to write the general hydrodynamic equation Eq. 23 explicitly depending on the pressures. Hence we shall find out how the tensors $\mathbf{Q}_n$ can be expressed in terms of the pressures $\mathbf{P}_n$.

Let us note a particular case of Eq. 22. For $n = 0$, bearing in mind the boundary condition Eq. 11, we get

$$\mathbf{Q}_1 = -\int_{\mathbf{u}} \nabla_{\mathbf{u}}\phi \, d\mathbf{u} = \phi|_{\mathbf{u}} = \mathbf{0} \tag{24}$$

For $n = 1$, the tensor product $(\mathbf{u})^n \otimes \nabla_{\mathbf{u}}\phi$ within the integral of Eq. 22 verifies, in components,

$$u_i \frac{\partial \phi}{\partial u_j} = \frac{\partial (u_i \phi)}{\partial u_j} - \delta_{ij}\, \phi \tag{25}$$

being δ_{ij} the Kronecker delta, and for $n \geq 2$,

$$\begin{aligned} u_{i_1} \ldots u_{i_n} \frac{\partial \phi}{\partial u_{i_{n+1}}} &= \frac{\partial (u_{i_1} \ldots u_{i_n} \phi)}{\partial u_{i_{n+1}}} - \\ &- \left(\delta_{i_1 i_{n+1}}\, u_{i_2} \ldots u_{i_n} + \ldots + \delta_{i_j i_{n+1}}\, u_{i_1} \ldots \widehat{u_{i_j}} \ldots u_{i_n} + \ldots + \delta_{i_n i_{n+1}}\, u_{i_1} \ldots u_{i_{n-1}}\right) \phi \end{aligned} \tag{26}$$

where the hat remarks the omitted factors.

Then, the tensor $\mathbf{Q}_{n+1}$ can be evaluated by integrating Eq. 25 and Eq. 26. The conditions of Eq. 11 are once more applied over the integration boundary, so that the first term on the right-hand side of Eq. 26, when integrating over $u_{i_{n+1}}$, yields

$$\int_{u_{i_{n+1}}} \frac{\partial (u_{i_1} \ldots u_{i_n} \phi)}{\partial u_{i_{n+1}}}\, du_{i_{n+1}} = u_{i_1} \ldots u_{i_n}\, \phi|_{u_{i_{n+1}}} = 0 \tag{27}$$

Hence, the tensor $\mathbf{Q}_{n+1}$ is obtained by integrating only the remaining terms, and by taking into account Eq. 8 and Eq. 9.

Thus, for $n = 1$ we are led to

$$(\mathbf{Q}_2)_{ij} = \delta_{ij}\, \mathbf{P}_0 \tag{28}$$

and for $n \geq 2$, we get the following expression depending on the pressures,

$$(\mathbf{Q}_{n+1})_{i_1 \ldots i_{n+1}} = \delta_{i_1 i_{n+1}}\, P_{i_2 \ldots i_n} + \ldots + \delta_{i_j i_{n+1}}\, P_{i_1 \ldots \widehat{i_j} \ldots i_n} + \ldots + \delta_{i_n i_{n+1}}\, P_{i_1 \ldots i_{n-1}} \tag{29}$$

The foregoing relationships will be used to write both terms in Eq. 23, which involve the tensor $\mathbf{Q}_{n+1}$. One of the terms contains a single dot product of this tensor with a vector, namely $\mathbf{a} \cdot \mathbf{Q}_{n+1}$. Hence, by applying Eq. 29, we get

$$
\begin{aligned}
(\mathbf{a}\cdot\mathbf{Q}_{n+1})_{i_1\ldots i_n} &= a_{i_{n+1}}\left(\delta_{i_1 i_{n+1}}P_{i_2\ldots i_n}+\ldots+\delta_{i_j i_{n+1}}P_{i_1\ldots\widehat{i_j}\ldots i_n}+\ldots+\delta_{i_n i_{n+1}}P_{i_1\ldots i_{n-1}}\right)= \\
&= a_{i_1}P_{i_2\ldots i_n}+\ldots+a_{i_j}P_{i_1\ldots\widehat{i_j}\ldots i_n}+\ldots+a_{i_n}P_{i_1\ldots i_{n-1}}
\end{aligned}
\tag{30}
$$

where Einstein's summation convention is used. The result is the symmetrised tensor product in regard to permutations of indices, $\mathcal{S}(\mathbf{a}\otimes\mathbf{P}_{n-1})$, which will be represented according to the following notation[3] (Cubarsi 1992),

$$
\mathcal{S}(\mathbf{a}\otimes\mathbf{P}_{n-1}) = n\,\mathbf{a}\star\mathbf{P}_{n-1},\ n\geq 1 \tag{31}
$$

so that the number of summation terms, which are needed in order to symmetrise the tensor product, is explicitly written. Notice that the only non-standard notation used in the current work is such a star product, which is defined in the footnote, since it worthy simplifies the forthcoming formulas.
Hence, Eq. 30 now stands for

$$
\mathbf{a}\cdot\mathbf{Q}_{n+1} = n\,\mathbf{a}\star\mathbf{P}_{n-1},\ n\geq 1 \tag{32}
$$

and, therefore,

$$
\left(\frac{d\mathbf{v}}{dt}+\nabla_{\mathbf{r}}\mathcal{U}\right)\cdot\mathbf{Q}_{n+1} = n\,\left(\frac{d\mathbf{v}}{dt}+\nabla_{\mathbf{r}}\mathcal{U}\right)\star\mathbf{P}_{n-1},\ n\geq 1 \tag{33}
$$

For the particular case $n=0$, by taking into account Eq. 24, the relation $\left(\frac{d\mathbf{v}}{dt}+\nabla_{\mathbf{r}}\mathcal{U}\right)\cdot\mathbf{Q}_1=0$ is fulfilled, which means, from an algebraic viewpoint, that Eq. 33 is also valid for $n\geq 0$, since the factor n appearing in Eq. 33 would make null the equality, even though $\mathbf{P}_{-1}$ is not defined.
In a similar way, Eq. 23 will be calculated for the double product $\nabla_{\mathbf{r}}\mathbf{v}:\mathbf{Q}_{n+2}$. Indeed, for $n=0$ Eq. 28 simply leads to

$$
\nabla_{\mathbf{r}}\mathbf{v}:\mathbf{Q}_2 = (\nabla_{\mathbf{r}}\cdot\mathbf{v})\,\mathbf{P}_0 \tag{34}
$$

while for $n\geq 1$, according to Eq. 29, it can be written in components as follows

[3] In general, if $\mathbf{A}_m$ and $\mathbf{B}_n$ are two m- and n-rank symmetric tensors, we can define the tensor $\mathbf{A}_m\star\mathbf{B}_n$ as the obtained by symmetrising the tensor product $\mathbf{A}_m\otimes\mathbf{B}_n$, and by normalising then with respect to the number of summation terms, T. The result is a $(m+n)$-rank symmetric tensor, whose components are

$$
(\mathbf{A}_m\star\mathbf{B}_n)_{i_1 i_2\ldots i_{m+n}} = \frac{1}{T}\,\mathcal{S}\,(\mathbf{A}_m\otimes\mathbf{B}_n)_{i_1 i_2\ldots i_{m+n}} = \frac{1}{T}\sum_{\substack{\alpha i_1<\ldots<\alpha i_m \\ \alpha i_{m+1}<\ldots<\alpha i_{m+n}}} A_{\alpha i_1\ldots\alpha i_m}B_{\alpha i_{m+1}\ldots\alpha i_{m+n}}
$$

where α belongs to the symmetric group $\mathrm{S}(m+n)$. If both tensors are different ones, then $T=\frac{(m+n)!}{n!m!}$. Notice that, in particular, if $\mathbf{A}_m=\mathbf{B}_n$ the number of summation terms is $T=\dfrac{(2n)!}{2!n!n!}$, and for the symmetric tensor product $\mathcal{S}(\bigotimes^k\mathbf{A}_n)$ the number of terms is $T=\dfrac{(kn)!}{k!(n!)^k}$.

$$(\nabla_{\mathbf{r}}\mathbf{v} : \mathbf{Q}_{n+2})_{i_1\ldots i_n} =$$

$$= \frac{\partial v_{i_{n+2}}}{\partial r_{i_{n+1}}} \left(\delta_{i_1 i_{n+2}} P_{i_2\ldots i_{n+1}} + \ldots + \delta_{i_j i_{n+2}} P_{i_1\ldots \widehat{i_j} \ldots i_{n+1}} + \ldots + \delta_{i_n i_{n+2}} P_{i_1 \ldots \widehat{i_n} i_{n+1}} + \delta_{i_{n+1} i_{n+2}} P_{i_1 \ldots i_n} \right) =$$

$$= \frac{\partial v_{i_1}}{\partial r_{i_{n+1}}} P_{i_2\ldots i_{n+1}} + \ldots + \frac{\partial v_{i_j}}{\partial r_{i_{n+1}}} P_{i_1\ldots \widehat{i_j} \ldots i_{n+1}} + \ldots + \frac{\partial v_{i_n}}{\partial r_{i_{n+1}}} P_{i_1 \ldots \widehat{i_n} i_{n+1}} + \frac{\partial v_{i_{n+1}}}{\partial r_{i_{n+1}}} P_{i_1 \ldots i_n} \tag{35}$$

The last term of the expression above is equivalent to $(\nabla_{\mathbf{r}} \cdot \mathbf{v})\, \mathbf{P}_n$, while the first n-terms are the components of the symmetrised tensor product $n\,(\mathbf{P}_n \cdot \nabla_{\mathbf{r}}) \star \mathbf{v}$. Thus, for $n \geq 1$, the foregoing equation can be written as

$$\nabla_{\mathbf{r}}\mathbf{v} : \mathbf{Q}_{n+2} = n\,(\mathbf{P}_n \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} + (\nabla_{\mathbf{r}} \cdot \mathbf{v})\, \mathbf{P}_n \tag{36}$$

Once more such a relation can formally be used also for $n = 0$, since, even though the dot product $(\mathbf{P}_0 \cdot \nabla_{\mathbf{r}})$ which would appear in the first term of the right-hand side is not defined, it would become null being multiplied by n.

Finally, by substitution of Eq. 33 and Eq. 36 into Eq. 23, and taking into account the definition of the material derivative, the general expression for an arbitrary n^{th}-order hydrodynamic equation becomes

$$\frac{\partial \mathbf{P}_n}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\,\mathbf{P}_n + n\,\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\mathbf{v} + \nabla_{\mathbf{r}}\mathcal{U}\right)\star\mathbf{P}_{n-1} + \nabla_{\mathbf{r}}\cdot\mathbf{P}_{n+1} + n\,(\mathbf{P}_n\cdot\nabla_{\mathbf{r}})\star\mathbf{v} + (\nabla_{\mathbf{r}}\cdot\mathbf{v})\,\mathbf{P}_n = (\mathbf{0})^n \tag{37}$$

Therefore, for each n, the foregoing equation, which is written in terms of the generalised tensor of pressures $\mathbf{P}_n$, is explicitly providing its conservation law.

5 Moment equations

The lowest order hydrodynamic equations are generally used, together with some additional hypotheses like axisymmetry, steadiness, incompressible flow, etc., and together with some closure assumptions related to diffusion (e.g. by neglecting off-diagonal second moments), conductivity (e.g. by neglecting third moments and higher odd-order moments), etc., in order to estimate either kinematic parameters of the local stellar populations, or the local stellar density, similarly to the earliest works by Jeans (1922) and Oort (1932), or like more recent works by Bahcall (1984a,b), Jarvis & Freeman (1985), van der Marel (1991), Famaey & Dejonghe (2003), most of them by using also the Poisson equation for self-gravitating systems or Stäckel models in order to close the system of equations. For the lowest orders, it is easy to give a physical interpretation of the stellar hydrodynamic equations by comparing them with the ones of fluid dynamics. Thus, for $n = 0$, bearing in mind that $\mathbf{P}_0 = N$ and $\mathbf{P}_1 = \mathbf{0}$, the continuity equation can be written in its transfer form, and by using the material derivative Eq. 16, as

$$\frac{d \ln N}{dt} = -\nabla_{\mathbf{r}} \cdot \mathbf{v} \tag{38}$$

Hence, since $\frac{d\ln N}{dt} = -\frac{d\ln N^{-1}}{dt}$, the divergence of the mean velocity yields the fractional time rate of change of the density N, as well as of the specific volume N^{-1}.
For $n = 1$ the equation of momentum transfer, which is usually referred as Jeans equation, is

$$\frac{d\mathbf{v}}{dt} = -\nabla_{\mathbf{r}}\mathcal{U} - \frac{1}{N}\nabla_{\mathbf{r}} \cdot \mathbf{P}_2 \tag{39}$$

and it is equivalent to the Navier-Stokes equation of fluid dynamics. Thus, the acceleration of the centroid is partially due to the force coming from the potential (per unit mass), and partially due to the force coming from relative pressure variations, that is, from the surface forces applied on a volume element. Let us remember that the second-order pressure tensor, which is also known as the comoving stress tensor, gives account, in its diagonal elements, of the normal stresses, and, in its non-diagonal elements, of the tangential stresses, which are associated with viscosity and diffusion effects.
For $n = 2$, Eq. 37 yields

$$\frac{d\mathbf{P}_2}{dt} = -(\nabla_{\mathbf{r}} \cdot \mathbf{v})\,\mathbf{P}_2 - 2\,(\mathbf{P}_2 \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} - \nabla_{\mathbf{r}} \cdot \mathbf{P}_3 \tag{40}$$

The first and second terms in the right-hand side of Eq. 40 are related to the rate of strain. The first one contains the divergence of the mean velocity, or volumetric rate of strain, while the second term contains the symmetric tensor $\nabla_{\mathbf{r}} \star \mathbf{v}$, which is the shear rate of strain. Although the trace of the above tensor equation provides the law for energy transfer, the six scalar equations involved in it give account of the work balance along the different transfer directions. Thus, according to the usual interpretation from fluid dynamics, the variation of internal energy in each direction is partially due to the work coming from the specific volume variation (first right-hand term), to the viscous dissipation through the surface of a volume element (second right-hand term), and from the heat added through conduction (third right-hand term).
In a general way, the transfer of the n^{th}-order pressure, Eq. 23, which we had obtained in §3, can be interpreted by taking into account Eq. 39, and by writing it in the following form,

$$\frac{d\mathbf{P}_n}{dt} = -\mathbf{Q}_{n+1} \cdot \frac{\nabla_{\mathbf{r}} \cdot \mathbf{P}_2}{N} - \mathbf{Q}_{n+2} : \nabla_{\mathbf{r}}\mathbf{v} - \nabla_{\mathbf{r}} \cdot \mathbf{P}_{n+1} \tag{41}$$

Thus the changes in pressure $\mathbf{P}_n$ are explained from a first term linearly depending on the rate of stress variation per unit mass, through the tensor $\mathbf{Q}_{n+1}$, from a second term linearly depending on the velocity gradient, through the tensor $\mathbf{Q}_{n+2}$, and from a third term giving account of the nearest higher-order pressure variation $\mathbf{P}_{n+1}$.
However, the general hydrodynamic equation Eq. 37 is not explicitly written in terms of data actually available from stellar velocity catalogues, since the pressures obviously depends on the stellar density, according to Eq. 9, while the central velocity moments can be directly computable from large stellar samples. By working from the velocity moments, together with the hydrodynamic equations and some appropriate closure conditions, it is possible to estimate or to model either the stellar density, the velocity distribution, or the potential function (e.g. Cuddeford & Amendt 1991). Similarly, numerical approaches and simulations by using the moment equations have not to be restricted either to orders $n < 2$, or to the assumption of vanishing odd-order moments (e.g. Vorobyov & Theis 2006).
For that reason, the general n^{th}-order equation will be expressed in terms of the central moments $\mathbf{M}_n$. The continuity equation, Eq. 38, does not need to be re-written, while for $n \geq 1$, Eq. 38 can be used together with Eq. 9 to re-write the general relation Eq. 37 in the following form

$$\frac{\partial \mathbf{M}_n}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}\,\mathbf{M}_n+n\,\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}\mathbf{v}+\nabla_{\mathbf{r}}\mathcal{U}\right)\star\mathbf{M}_{n-1}+(\nabla_{\mathbf{r}}\ln N+\nabla_{\mathbf{r}})\cdot\mathbf{M}_{n+1}+n\,(\mathbf{M}_n\cdot\nabla_{\mathbf{r}})\star\mathbf{v}=(\mathbf{0})^n \tag{42}$$

Hence, for $n=1$, the momentum equation can be expressed as follows

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}\mathbf{v}+\nabla_{\mathbf{r}}\mathcal{U}=-(\nabla_{\mathbf{r}}\ln N+\nabla_{\mathbf{r}})\cdot\mathbf{M}_2 \tag{43}$$

and for $n=2$, since $\mathbf{M}_1=\mathbf{0}$, we have

$$\frac{\partial \mathbf{M}_2}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}\,\mathbf{M}_2+(\nabla_{\mathbf{r}}\ln N+\nabla_{\mathbf{r}})\cdot\mathbf{M}_3+2\,(\mathbf{M}_2\cdot\nabla_{\mathbf{r}})\star\mathbf{v}=(\mathbf{0})^2 \tag{44}$$

In general, Eq. 43 may be introduced into the higher-order equations to replace the terms depending on the potential function, so that they remain written in terms of the comoving moments. Then, for $n\geq 2$ we have

$$\frac{\partial \mathbf{M}_n}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}\,\mathbf{M}_n-n\,\left[(\nabla_{\mathbf{r}}\ln N+\nabla_{\mathbf{r}})\cdot\mathbf{M}_2\right]\star\mathbf{M}_{n-1}+(\nabla_{\mathbf{r}}\ln N+\nabla_{\mathbf{r}})\cdot\mathbf{M}_{n+1}+n\,(\mathbf{M}_n\cdot\nabla_{\mathbf{r}})\star\mathbf{v}=(\mathbf{0})^n \tag{45}$$

which is the general expression[4] giving the contributing terms to the conservation of the n^{th}-order moment.

Nevertheless, let us remember the typical situation we are led when working with hydrodynamic equations. The equations Eq. 38 and Eq. 43, for $n=0$ and $n=1$, contain four different scalar equations, which involve a set of eleven unknown scalar functions, namely $N,\mathcal{U},\mathbf{v}$ and the symmetric tensor $\mathbf{M}_2$. It is well known that, even in the case of taking into account higher-order equations the system remains always open, since by picking up the m^{th}-equation, which contains $\binom{m+2}{2}$ scalar equations, we are also introducing as many as $\binom{m+3}{2}$ new unknowns, which are the different components of the tensor $\mathbf{M}_{m+1}$.

In most cases, the lowest order equations are used under some particular assumptions to reduce the number of unknowns. For example, by assuming the epicyclic approach, like in Oort (1965), by assuming axial symmetry (e.g. Vandervoort 1975), or by taking a velocity distribution function depending on specific isolation integrals of the star motion (e.g. Jarvis & Freeman 1985). In such a way, some constraint relationships for the central moments may be reached (e.g. van der Marel 1991). Now, for specific velocity distributions and working from the general moment equation, the closure conditions could be studied in a more general way.

[4]The component $i_1\ldots i_n$ for $n\geq 2$ of the equation stands for

$$\begin{aligned}
&\frac{\partial M_{i_1\ldots i_n}}{\partial t}+v_\alpha\frac{\partial M_{i_1\ldots i_n}}{\partial r_\alpha}-\left(\frac{\partial M_{\alpha i_1}}{\partial r_\alpha}M_{i_2\ldots i_n}+\frac{\partial M_{\alpha i_2}}{\partial r_\alpha}M_{i_1\widehat{i_2}\ldots i_n}+\ldots+\frac{\partial M_{\alpha i_n}}{\partial r_\alpha}M_{i_1\ldots i_{n-1}}\right)+\\
&+\frac{\partial \ln N}{\partial r_\alpha}\left(M_{\alpha i_1\ldots i_n}-M_{\alpha i_1}M_{i_2\ldots i_n}-M_{\alpha i_2}M_{i_1\widehat{i_2}\ldots i_n}-\ldots-M_{\alpha i_n}M_{i_1\ldots i_{n-1}}\right)+\\
&+\frac{\partial M_{\alpha i_1\ldots i_n}}{\partial r_\alpha}+\frac{\partial v_{i_1}}{\partial r_\alpha}M_{\alpha i_2\ldots i_n}+\frac{\partial v_{i_2}}{\partial r_\alpha}M_{\alpha i_1\widehat{i_2}\ldots i_n}+\ldots+\frac{\partial v_{i_n}}{\partial r_\alpha}M_{\alpha i_1\ldots i_{n-1}}=0
\end{aligned} \tag{46}$$

6 Closure example

Although the study of closure conditions is beyond the scope of the present work, we shall see a simple example, as a basis to future works, of how to use the n^{th}-order general expression, Eq. 45, to find out the closure conditions in terms of the velocity distribution statistics.

Let us assume the simplest case of an isothermal velocity distribution of Maxwell type in the residual velocities, according to the Maxwell-Boltzmann law, which represents a system with the more basic thermal equilibrium. A more general case for Schwarzschild and generalised Schwarzschild distributions will be studied in detail in a forthcoming paper. Then for Eq. 12 we have

$$\phi(t, \mathbf{r}, \mathbf{u}) = e^{-\frac{1}{2}Q}, \quad Q = \mu^{-1}\,\mathbf{u}^{\mathrm{T}} \cdot \mathbf{u} \tag{47}$$

where Q is a quadratic, positive definite form, and $\mu(t, \mathbf{r})$ is a continuous and differentiable function in both arguments, which gives account of the variance of the distribution.

This is a well known particular case of an ellipsoidal dynamical model (Chandrasekhar 1942, p.136), which can be solved by substitution of Eq. 47 in the collisionless Boltzmann equation Eq. 2, or directly in his form Eq. 15. By this way, the dynamical model is determined from a finite set of equations, but we may wonder about how is it related to the infinite hierarchy of hydrodynamic equations. Although we know that all the hydrodynamic equations are fulfilled, we may guess that there is a finite subset of hydrodynamic equations which are strictly equivalent to the collisionless Boltzmann equation. Then, which are the orders of these equations? Why and which are the redundant equations? Can we explicitly write the conditions that make them redundant? In other words, which are the closure conditions? In general, the answers to the foregoing questions vary depending on the form of the velocity distribution function. This example claims to be the seed of more complete and general works on closure conditions.

The Maxwellian distribution in the residual velocities, since it is a totally isotropic distribution, has all the odd-order central moments null. Its symmetric even-order moments are given through isotropic tensors accordingly to

$$\mathbf{M}_{2n} = \mathcal{S}(\otimes^n \mathbf{I}_2\,\mu) = C_n\,\mu^n\,\overbrace{\mathbf{I}_2 \star \cdots \star \mathbf{I}_2}^{n}, \quad C_n = \frac{(2n)!}{2^n n!} \tag{48}$$

where $\mathbf{I}_2$ is representing the second-rank identity tensor. Since $C_{n+1} = (2n+1)\,C_n$, it is easy to prove the relation

$$\mathbf{M}_{2n+2} = (2n+1)\,\mathbf{M}_{2n} \star \mathbf{I}_2\,\mu \tag{49}$$

For the even-order equations, $n = 2k$ and $k \geq 1$, bearing in mind that the odd-moments are null, Eq. 45 becomes

$$\frac{\partial \mathbf{M}_{2k}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}}\,\mathbf{M}_{2k} + 2k\,(\mathbf{M}_{2k} \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} = (\mathbf{0})^{2k} \tag{50}$$

which, by substitution of moment expressions Eq. 48 and Eq. 49, easily reduces to

$$\mathbf{M}_{2k-2} \star \left[\frac{\partial(\mathbf{I}_2\,\mu)}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}}\,(\mathbf{I}_2\,\mu) + 2\,(\mathbf{I}_2\,\mu \cdot \nabla_{\mathbf{r}}) \star \mathbf{v}\right] = (\mathbf{0})^{2k} \tag{51}$$

Thus, since $\mathbf{M}_{2k-2}$ never vanishes, we conclude that all the even-order equations, $n \geq 2$, are reduced to the second-order equation

$$\mathbf{I}_2\left(\frac{\partial\mu}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}\mu\right)+2\,\mu\nabla_{\mathbf{r}}\star\mathbf{v}=(\mathbf{0})^2 \tag{52}$$

Therefore, such a relationship provides a closure condition in terms of the mean velocity, and of the velocity variance μ.
In a similar way, for the odd-order equations, $n=2k+1$, $k\geq 1$, provided that the odd-order moments are null, from Eq. 45 we can write

$$-(2k+1)\;\left[(\nabla_{\mathbf{r}}\ln N+\nabla_{\mathbf{r}})\cdot\mathbf{M}_2)\right]\star\mathbf{M}_{2k}+(\nabla_{\mathbf{r}}\ln N+\nabla_{\mathbf{r}})\cdot\mathbf{M}_{2k+2}=(\mathbf{0})^{2k+1} \tag{53}$$

Then, by substitution of moment expressions Eq. 48 and Eq. 49, we have

$$\begin{gathered}-(2k+1)\;\mu\nabla_{\mathbf{r}}\ln N\star\mathbf{M}_{2k}-(2k+1)\;\nabla_{\mathbf{r}}\mu\star\mathbf{M}_{2k}+\\+(2k+1)\,\mu\nabla_{\mathbf{r}}\ln N\cdot(\mathbf{I}_2\star\mathbf{M}_{2k})+(2k+1)\,\nabla_{\mathbf{r}}\cdot(\mu\mathbf{I}_2\star\mathbf{M}_{2k})=(\mathbf{0})^{2k+1}\end{gathered} \tag{54}$$

In addition, from Eq. 48 we can obtain the following identity,

$$\nabla_{\mathbf{r}}\cdot(\mu\mathbf{I}_2\star\mathbf{M}_{2k})=\nabla_{\mathbf{r}}\mu\star\mathbf{M}_{2k}+k\,\mu\nabla_{\mathbf{r}}\star\mathbf{M}_{2k} \tag{55}$$

which allow us to simplify Eq. 54, by leading to

$$\nabla_{\mathbf{r}}\star\mathbf{M}_{2k}=(\mathbf{0})^{2k+1} \tag{56}$$

This expression stands for all the odd-order equations, $n=2k+1\geq 3$.
Finally, by taking into account Eq. 49, since $\mathbf{M}_{2k-2}$ is always non-null for $k\geq 1$, then Eq. 56 can be reduced to the third-order equation

$$\mathbf{I}_2\star\nabla_{\mathbf{r}}\mu=(\mathbf{0})^3 \tag{57}$$

Hence, such a relation provides another closure condition in terms of the velocity variance μ. In other words, for a velocity distribution of Maxwell type, if the closure conditions given by Eq. 52 and Eq. 57 are satisfied, then all the moment equations are reduced to the four equations of orders $n=0,1,2,3$, which are a set of twenty scalar equations, and they are then equivalent to the collisionless Boltzmann equation.
Thus, such a simple example shows how to find out dependences of higher-order hydrodynamic equations on the lower-order ones, for a specific velocity distribution. Although this model has no interesting physical implications, it can be easily solved from the foregoing equations. Thus, from Eq. 57, we can find that μ is not a function of $\mathbf{r}$. Also, from Eq. 52, the mean velocity components can be obtained[5]. Henceforth, from the equations corresponding to $n=0$ and $n=1$, and depending on the specific time-dependence or symmetry assumptions for the stellar system in general, or for the potential function in particular, the remaining unknowns can be determined in every situation.

[5]In components, Eq. 57 becomes $\delta_{ij}\frac{\partial\mu}{\partial r_k}+\delta_{ik}\frac{\partial\mu}{\partial r_j}+\delta_{jk}\frac{\partial\mu}{\partial r_i}=0$, which implies $\frac{\partial\mu}{\partial r_i}=0$ for all i. Then Eq. 52 reduces to $\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}=-\frac{\partial\ln\mu}{\partial t}\delta_{ij}\equiv 2\dot{\kappa}\delta_{ij}$. Thus, the mean velocity satisfies $\frac{\partial v_1}{\partial r_1}=\frac{\partial v_2}{\partial r_2}=\frac{\partial v_3}{\partial r_3}=\dot{\kappa}$ and $\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}=0$ for $i\neq j$. Hence, the shear strain rate is null, and there is some volumetric rate of strain only for a non-stationary system. By solving these equations, we easily obtain a mean velocity in the form $\mathbf{v}=\dot{\kappa}\,\mathbf{r}+\omega\wedge\mathbf{r}+\tau$, which means that the centroid moves accordingly to a radial expansion or contraction, in the case of a non stationary system, and rotates like a rigid body, with angular velocity ω, in addition to an arbitrary translation τ.

7 Concluding remarks

The exact and full expression of the stellar hydrodynamic equations for any arbitrary order was deduced from the collisionless Boltzmann equation. It was written without any restrictive assumptions, like those of steady-state system, axial symmetry, galactic plane of symmetry, pure rotating system, vanishing odd-order moments, etc., so that it can be used, for example, under some of these hypotheses to test analytical dependences of the phase space density function on the integrals of motion, or in its complete form to carry out numerical simulations about either the velocity distribution or stellar density variations.

It is worth noting that most of the aforementioned hypotheses are not already valid in the solar neighbourhood, as it can be shown working from *HIPPARCOS* catalogue. Alcobé & Cubarsi (2005) and Cubarsi & Alcobé (2004, 2006) discussed how the local thin disc was clearly non-axisymmetric, with a non-vanishing vertex deviation, whereas increasing nested subsamples of thick disc stars showed a trend to axisymmetry. Similarly, they found that the local thin disc was not in steady-state, which was related to its net radial velocity towards the galactic centre, whereas thick disc stars did showed a trend to steady-state. Therefore, the moment equations have to be actually used in their complete form, at least for Galactic disc analysis.

On the other hand, the general expression of moment equations may be also useful to study closure conditions, which are associated with specific velocity distributions, in order to reduce the infinite hierarchy of equations and unknowns to a finite number of them, so that a feasible dynamical model can be available. Obviously, when working with Jeans equation alone, or whatever finite set of hydrodynamic equations, the collisionless Boltzmann equation is not generally fulfilled. It is an interesting mathematical problem to study how the system of equations can be closed, and which are then the conditions over the the velocity distribution function in order to exactly fulfil Boltzmann equation. Thus, by using a similar procedure as in the case example, some more general and interesting cases, like those of Gaussian and ellipsoidal distributions, as well as finite mixtures of them, can be studied in forthcoming works.

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